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Restricted Centers in Subalgebras of $C(X)$

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1. INTRODUCTION

Let X be a real normed linear space, G be a subset of X, and F be a bounded subset of X. The purpose of this paper is to study the following minimization problem which is interesting since it is a generalization of the best approximation problem:

$$\inf \sup\{\| g - f \| : f \in F\} \equiv R_G(F) \tag{1.1}$$

attained, where the infimum is taken over all $g \in G$?

Letting $E_G(F)$ denote the set of points of G where the infimum is attained, we say that G has the restricted center property if $E_G(F) \neq \varnothing$ for all bounded sets $F \subset X$. Restricted centers are a natural generalization of Chebyshev centers and the arguments used throughout this paper are quite reminiscent of those employed in Chebyshev center theory (see for example [1]). In this paper, it is shown that closed subalgebras of $C(X)$, where X is a compact Hausdorff space, have the restricted center property. This generalizes the known fact that closed subalgebras in $C(X)$ are proximinal (cf. [5, p. 124]). In addition, a stability result on the sets of centers is obtained and necessary and sufficient conditions for a certain class of bounded sets to have a unique restricted center are given.

2. EXISTENCE

Our aim here is to prove:

THEOREM 1. *Every closed subalgebra $\mathscr{A}$ of $C(X)$ has the restricted center property.*

We need the following notation and lemmas: X and Y are compact Hausdorff spaces; $C(X)$ is the space of real valued continuous functions on X; $\pi: X \to Y$ is a continuous surjection; $C(X/\pi) = \{f \in C(X) : f = g \circ \pi,$

$g \in C(Y)\}$; $C_\lambda(X/\pi) = \{f \in C(X) : f = g \circ \pi, g \in C(Y), g(\lambda) = 0\}$; lsc$(X)$ is the set of real valued lower semicontinuous functions on X; and usc(X) is the set of real valued upper semicontinuous functions on X.

We now state four lemmas which are either evident or well-known.

LEMMA 1. *Let G and H be two sets such that $G \supset H$. Then $R_G(F) \leqslant R_H(F)$, for any bounded set F.*

LEMMA 2. *Let X be a compact space and let $\mathscr{A}$ be a closed subalgebra of $C(X)$. Then there exists a continuous surjection π from X onto a compact space Y such that $\mathscr{A} = C_\lambda(X/\pi)$ and $\pi^{-1}(\lambda) = \{z \in X : f(z) = 0$ for all $f \in \mathscr{A}\}$.*

Proof. This is essentially contained in [4, p. 191] and [5, p. 122].

LEMMA 3. *Let X, Y be compact, $\theta : X \to Y$ a continuous surjection and $f \in lsc(X)$. Then $f^{\downarrow}(y) = \inf\{f(x) : x \in \theta^{-1}(y)\}$ is in $lsc(Y)$.*

Proof. The above was proved for $f \in C(X)$ by Semadeni (cf. [5, p. 124]). An examination of his proof shows that the lemma holds for $f \in lsc(X)$.

LEMMA 4. *Let $h \in lsc(X)$, let Ω be a closed subset of X, and let $f \in lsc(\Omega)$ such that $f(y) \leqslant h(y)$ for $y \in \Omega$. Then the function*

$$\begin{aligned} h'(x) &= f(x) \quad \text{if} \quad x \in \Omega \\ &= h(x) \quad \text{if} \quad x \in X \backslash \Omega \end{aligned}$$

is also in $lsc(X)$.

Proof of Theorem 1. Let B be any bounded set in $C(X)$ and define

$$\begin{aligned} m(t) &= \sup\{y(t) : y \in B\} \\ u(t) &= \lim\sup\{m(\tau) : \tau \to t\} \\ n(t) &= \inf\{y(t) : y \in B\} \\ v(t) &= \lim\inf\{n(\tau) : \tau \to t\}. \end{aligned}$$

It is easy to check that $u(v)$ is in $usc(X)$ $(lsc(X))$. By Lemma 2, $\mathscr{A} = C_\lambda(X/\pi)$ for some compact Y and continuous surjection $\pi : X \to Y$. Define

$$2R(y) = \sup\{| u(x) - v(x')| : x, x' \in \pi^{-1}(y)\},$$

and let

$$R' = \max\{R(y) : y \in Y\}.$$

Define $R = \max\{\| u - v \|/2, \| u|_Z \|, \| v|_Z \|, R'\}$ where $Z = \{x \in X : f(x) = 0$ for all $f \in \mathscr{A}\}$ and $\| u|_Z \| = \max | u(t)|$, $t \in Z$. If $Z = \varnothing$, set $\| u|_Z \| = \| v|_Z \| = 0$.

We now show $R_{\mathscr{A}}(B) \geqslant R$. An argument used in [1, 2] shows that $\| u - v \|/2 \leqslant R_{C(X)}(B)$. Thus, $\| u - v \|/2 \leqslant R_{C(X)}(B) \leqslant R_{\mathscr{A}}(B)$ by Lemma 1. For all $f \in \mathscr{A}$, $f|_Z = 0$; hence $R_{\mathscr{A}}(B) \geqslant \max\{\| u|_Z \|, \| v|_Z \|\}$. Also, since for all $f \in \mathscr{A}$ and $y \in Y$, $f|_{\pi^{-1}(y)}$ is a constant, clearly $R_{\mathscr{A}}(B) \geqslant R'$. Thus $R_{\mathscr{A}}(B) \geqslant R$.

Now suppose that there exists a $y' \in \mathscr{A}$ satisfying $u(t) - R \leqslant y'(t) \leqslant v(t) + R$ for all $t \in X$. Then since for all $x \in B$, $v \leqslant x \leqslant u$, we have

$$x - R \leqslant y' \leqslant x + R.$$

Hence, $\| x - y' \| \leqslant R$ for all $x \in B$ and thus

$$R_{\mathscr{A}}(B) = \inf \sup\{\| y - x \| : x \in B\} \geqslant R \geqslant \sup\{\| y' - x \| : x \in B\}$$

where the above infimum is taken over all $y \in \mathscr{A}$. Hence y' is a restricted center of B. It remains to show that such a y' exists.

Denote

$$v_1(y) = \inf\{v(x) + R : x \in \pi^{-1}(y), y \in Y\},$$
$$u_1(y) = \sup\{u(x) - R : x \in \pi^{-1}(y), y \in Y\}.$$

By Lemma 3, $v_1(u_1)$ is $lsc(Y)$ $(usc(Y))$. Now let

$$\begin{aligned} v_2(y) &= 0 \quad \text{if} \quad y = \lambda \\ &= v_1(y) \quad \text{otherwise} \\ u_2(y) &= 0 \quad \text{if} \quad y = \lambda \\ &= u_1(y) \quad \text{otherwise.} \end{aligned}$$

Since $v_l|_Z \geqslant 0 \geqslant u_l|_Z$, $v_2(u_2)$ is in $lsc(Y)$ $(usc(Y))$ by Lemma 4.

Hence, by the Dieudonné interposition theorem, there exists a $g \in C(Y)$ such that $u_2(y) \leqslant g(y) \leqslant v_2(y)$ for all $y \in Y$ and $\{g \in C(Y) : g(\lambda) = 0\}$. Set $y' = g \circ \pi$. Then $y' \in C_\lambda(X/\pi)$ and y' is a restricted center of B since for all $t \in X$,

$$(*) \quad u(t) - R \leqslant u_2 \circ \pi(t) \leqslant y'(t) \leqslant v_2 \circ \pi(t) \leqslant v(t) + R. \qquad \text{Q.E.D.}$$

Remark 1. It is easy to check that any $y' \in \mathscr{A}$ satisfying (*) is in $E_{\mathscr{A}}(B)$. A routine calculation shows that (*) characterizes $E_{\mathscr{A}}(B)$, i.e., $y' \in E_{\mathscr{A}}(B)$ iff $y' \in \mathscr{A}$ and y' satisfies (*).

3. Stability

In this section, we denote by $\phi(B_1, B_2)$ the distance in the Hausdorff metric between two bounded sets B_1 and B_2.

THEOREM 2. *Let B_1 and B_2 be arbitrary bounded sets in $C(X)$, and let $\mathscr{A}$ be a closed subalgebra of $C(X)$. If $\phi(B_1, B_2) \leqslant \epsilon$, then $\phi(E_{\mathscr{A}}(B_1), E_{\mathscr{A}}(B_2)) \leqslant 2\epsilon$.*

Since the proof of this theorem is very much like that of [2, Theorem 3], we omit the proof here.

4. UNIQUENESS

Our aim here is to get a uniqueness result for restricted centers of a bounded set $B \subset C(X)$ in terms of the functions $m(t)$ and $n(t)$ (notation is as in the proof of Theorem 1). Such a characterization for an arbitrary bounded set and closed subalgebra seems, to these authors, hopeless. Hence, we assume that the bounded set is contained in the closed subalgebra and that the subalgebra contains a unit. By the Gelfand–Naimark Theorem we are thus in the situation of trying to characterize those bounded sets in $C(X)$, X compact, having a unique center.

Under these assumptions, the centers for a bounded set B consist of

$$(**) \quad \{y: u(t) - R \leqslant y(t) \leqslant v(t) + R \quad \text{for all} \quad t \in X, R = \| u - v \|/2\}$$

where we are using the notation of Theorem 1. Our final assumption is that X be a perfect metric space. Under these hypotheses, Theorem 3 corrects an error in [2] (i.e., the hypothesis "$UR1$" should replace "removable" in [2, Theorem 2]).

DEFINITION 1. A function f is said to have an unremovable discontinuity at t_0 if f cannot be redefined at t_0 so as to make f continuous at t_0.

DEFINITION 2. A function f with an unremovable discontinuity at t_0 is said to have a point of unremovable discontinuity of type 1 ($UR1$) at t_0 if for each $\epsilon > 0$, there exists a neighborhood $\mathscr{N}_{t_0}$ of t_0 such that for all pairs of points (x, y) of continuity of f in $\mathscr{N}_{t_0} \times \mathscr{N}_{t_0}$, $|f(x) - f(y)| < \epsilon$.

DEFINITION 3. A function f with an unremovable discontinuity at t_0 is said to have a point of unremovable discontinuity of type 2 ($UR2$) at t_0 if the unremovable discontinuity at t_0 is not $UR1$.

EXAMPLE 1. Let $A = \{x_n\}_{n=1}^{\infty}$ where $x_n \in [0, 1]$ for all n and $x_n \to 0$. Let f be the characteristic function of A. Then f is $UR1$ at 0.

EXAMPLE 2. $\sin 1/x$ is $UR2$ at 0.

LEMMA 5. *m and n are UR2 at t_0 iff u and v are discontinuous at t_0.*

Proof. It suffices to consider only the case of n and v. ($\Rightarrow$) Assume n is $UR2$ at t_0. Now since n is in $usc(X)$, it is also a Baire 1 function and hence by Osgood's Theorem [3], n is continuous on a dense set in X. We claim v cannot be continuous at t_0. To see this, note that since n is $UR2$ at t_0, there exists an $\epsilon > 0$ such that for each neighborhood $\mathcal{N}_{t_0}$ of t_0 there exists a pair $(x, y) \in \mathcal{N}_{t_0} \times \mathcal{N}_{t_0}$ such that $|n(x) - n(y)| > \epsilon$ and n is continuous at x and y. Now v continuous at t_0 implies that there exists a $\mathcal{N}'_{t_0}$ such that $|v(t) - v(t_0)| < \epsilon/2$ for all $t \in \mathcal{N}'_{t_0}$. But since the points of continuity of n are dense in X and X is perfect, there exist distinct points x' and y' in $\mathcal{N}'_{t_0}$ such that $|n(x') - n(y')| > \epsilon$. At the points of continuity of n, $n(t) = v(t)$ implies $|v(x') - v(y')| > \epsilon$ implies either $|v(x') - v(t_0)|$ or $|v(y') - v(t_0)| > \epsilon/2$, a contradiction.

($\Leftarrow$) Now suppose v is discontinuous at t_0. Thus for some $\epsilon > 0$ and for each neighborhood $\mathcal{N}_{t_0}$, there exists a $t_1 \in \mathcal{N}_{t_0}$ such that $|v(t_1) - v(t_0)| > \epsilon$. Pick an $\mathcal{N}'_{t_0}$ "small enough" that $\{|\inf n(t) - v(t_0)| : t \in \mathcal{N}'_{t_0}\} < \epsilon/8$. Now since $v(t) = \lim\inf\{n(\tau), \tau \to t\}$, there exists a t_1' and a t_0' in $\mathcal{N}'_{t_0}$ such that $|v(t_1) - n(t_1')| < \epsilon/8 > |v(t_0(- n(t_0')|$. Since the points of continuity of n are dense in X and n is in $usc(X)$, there exists a t_1'' and t_0'' in $\mathcal{N}'_{t_0}$ that are points of continuity of n and $n(t_1'') < n(t_1') + \epsilon/8$ and $n(t_0'') < n(t_0') + \epsilon/8$.

Consequently, $|v(t_1) - n(t_1'')| < 3\epsilon/8 > |v(t_0) - n(t_0'')|$. Thus, $|n(t_1'') - n(t_0'')| > \epsilon/4$. Since $\mathcal{N}'_{t_0}$ was "small" but arbitrary, n is not $UR1$ at t_0, i.e., n is $UR2$ at t_0. Q.E.D.

THEOREM 3. *A bounded set B has a unique Chebyshev center in C(X) iff m(t) and n(t) are at most UR1 for all t, and, at the points of mutual continuity, the difference m(t) − n(t) is constant.*

Proof. ($\Rightarrow$) If m and/or n are $UR2$ at some t_0, then u and/or v are discontinuous at t_0 by the previous lemma. Since the semicontinuous functions are of different types, then $u(t) - v(t) \neq$ constant.

Suppose now m and n are at most $UR1$ for all t, but on the set K of points of mutual continuity $m(t) - n(t) \neq$ constant. In this case $u(t) = n(t)$ and $v(t) = n(t)$ on K, so $u(t) - v(t) \neq$ constant. Thus, in either case, we may find a point t_0 for which $v(t_0) + R > u(t_0) - R$. By Lemma 4, we can find two continuous functions y_1 and y_2 for which the following holds:

$$u(t) - R \leqslant y_i(t) \leqslant v(t) + R \qquad i = 1, 2$$
$$y_1(t_0) = v(t_0) + R;\ y_2(t_0) = u(t_0) - R.$$

In this way we see that the set B possesses at least two centers.

($\Leftarrow$) Suppose n and m are at most $UR1$ and the difference $m(t) - n(t)$ at the points of mutual continuity is constant. By Lemma 5, both u and v are continuous on X. Osgood's Theorem asserts that the points of discontinuity of m constitute a set of first category; the same is true of n. Thus, the points of mutual continuity of m and n are a dense set [3]. Since $m(t) = u(t)$ and $n(t) = v(t)$ at the points of mutual continuity, $u(t) - v(t) = 2R$, for all $t \in X$. Thus $u(t) - R = v(t) + R$ for all $t \in X$ and hence by (**), we get that the center is unique. Q.E.D.

Remark. Suppose X is not perfect. It then contains at least one isolated point. Thus a necessary condition for a bounded set $B \subset C(X)$ to have a unique center is that for all isolated points t_α, $u(t_\alpha) - R = v(t_\alpha) + R$. For if $u(t_\alpha) - R < v(t_\alpha) + R$ at an isolated point t_α, we may redefine a center to have any value in $[u(t_\alpha) - R, v(t_\alpha) + R]$ without affecting the continuity of the center.

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